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Demonstration of Soliton Solutions of DNLS Equation by Liouville Theorem

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In the inverse scattering transform (IST), the reflectionless Jost solutions are combined by their analytic properties in the complex spectrum parameter plane, and then can be shown to satisfy the two Lax equations indeed by Liouville theorem. So the corresponding soliton solutions certainly satisfy the nonlinear equation by compatibility condition. Especially the multi-soliton solutions of DNLS equation can be demonstrated in this way.

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1. INTRODUCTION

DNLS equation with vanishing boundary condition is

$$iu_t + u_{xx} + i(|u|^2u)_x = 0 \tag{1}$$

and its Lax pair is given by

$$L = \lambda(-i\lambda\sigma_3 + U), \qquad U = u\sigma_+ - \bar{u}\sigma_- \tag{2}$$

and

$$M = -i2\lambda^4\sigma_3 + 2\lambda^3 U - i\lambda^2 U^2\sigma_3 - \lambda(-U^3 + iU_x\sigma_3) \tag{3}$$

In order to solve it, Kaup and Newell introduce a new spectral parameter $\kappa = \lambda^{-1}$ and construct inverse scattering transform (IST) in complex κ-plane (Kaup and Newell, 1978a,b). Some other authors, Wadati *et al.* (1979), for example, do not recognize essentials of this procedure, and construct IST in complex λ-plane persistently. Though they can give explicit expression of a single-soliton solution equivalent to that of Kaup and Newell, the procedure of derivation and the final results are more complicated. On the other hand, it is hard to show N-soliton solution given by Wadati *et al.* equivalent to that given by Kaup and Newell.

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Recently, DNLS equation still attracts much attention, for example, there are some works for developing its perturbation theory (Chen and Yang, 2002; Kaup, 1990, 1991). So the soliton solutions of DNLS equation, especially in multi-soliton case, are needed to be demonstrated.

As is well known, the rightness of a soliton solution is verified finally by direct substitution. But for multi-soliton solution the direct substitution is only in principle, not in practice. In the case of nonlinear Schrödinger equation (NLS), its Jost solutions can be factorized as a product of Blaschiek's matrices in a recursive manner with the method of Darboux transformation matrix (or equivalently Riemann–Hilbert method with zeroes). As a result, if the single-soliton Jost solutions satisfy the Lax equations, the corresponding multi-soliton Jost solutions also do so in a recursive manner, and then the multi-soliton solutions certainly satisfy NLS equation by compatibility condition.

On the other hand, the demonstration of multi-soliton solutions in this way is also applicable to the results obtained by the IST method. Usually, the Jost solutions found by IST in reflectionless case are composed two in a 2×2 matrix form by same asymptotic behaviors, $\Psi = (\tilde{\psi},\ \psi)$ in $x \to \infty$, and $\Phi = (\phi,\ \tilde{\phi})$ in $x \to -\infty$. If one composes two matrices with the same analytic properties in a 2×2 matrix form, $\tilde{F} = (\tilde{\psi},\ \tilde{\phi})$ and $F = (\phi,\ \psi)$, they can be shown to be equivalent to those in the method of Darboux transformation matrix, or in Riemann–Hilbert method with zeros. Therefore, the Jost solutions obtained by IST method in reflectionless case satisfy also the Lax equations, and then the corresponding multi-soliton solutions definitely satisfy NLS equation by compatibility condition.

In this paper, we demonstrate multi-soliton solutions for DNLS equation given by Kaup–Newell in this way. With the matrix $\tilde{F}(x, t, \kappa)$ composed by two Jost solutions with the same analytic properties in complex κ-plane, we firstly introduce $\tilde{F}(x, t, \kappa)'$ that is shown to be its right inverse by Liouville theorem in complex analysis, and then to be its left inverse due to the case that they are 2×2 matrices. Secondly, we discuss the analytic properties of $\tilde{F}(x, t, \kappa)_x \tilde{F}(x, t, \kappa)'$ in whole κ-plane, and obtained its residues at $|\kappa| \to \infty$ and at $|\kappa| \to 0$. So the resulted expression of $\tilde{F}(x, t, \kappa)_x \tilde{F}(x, t, \kappa)'$ subtracting these two residues is analytic in the whole κ-plane and should be a constant by Liouville theorem. We thus show $\tilde{F}(x, t, \kappa)$ satisfy the first Lax equation. Finally, the analytic properties of $\tilde{F}(x, t, \kappa)_t \tilde{F}(x, t, \kappa)'$ are discussed with similar procedure and $\tilde{F}(x, t, \kappa)$ is found to satisfy the second Lax equation definitely. Therefore, the multi-soliton solutions obtained by compatibility condition of Jost solutions satisfy the DNLS equation.

2. JOST SOLUTIONS

The first Lax equation is

$$\partial_x F(x,\kappa) = L(x,\kappa)F(x,\kappa) \tag{4}$$

where $L(\kappa)$ is obtained from Equation (2) by simple replacement of λ by κ^{-1}. To do so, the first Lax pair $L(\kappa)$ is only singular at zero in the whole complex κ-plane, not like that $L(\lambda)$ in (2) is singular at infinity in the λ-plane, which is beneficial to solve DNLS equation by IST method.

The vanishing boundary condition of DNLS equation is, in the limit of $|x| \to \pm\infty$, $u \to 0$, and then $L \to -i\kappa^{-2}\sigma_3$. As a result, the free Jost solution is

$$E(x,\kappa) = e^{-i\kappa^{-2}x\sigma_3} \tag{5}$$

which expresses two independent solutions with two components as κ^{-2} is real. Correspondingly, the Jost solutions are defined as

$$\Psi(x,\kappa) = (\tilde{\psi}(x,\kappa),\ \psi(x,\kappa)) \to E(x,\kappa), \quad \text{as}\, x \to \infty \tag{6}$$

and

$$\Phi(x,\kappa) = (\phi(x,\kappa),\ \tilde{\phi}(x,\kappa)) \to E(x,\kappa), \quad \text{as}\, x \to -\infty \tag{7}$$

Then introducing monodramy matrix $T(\kappa)$, one has

$$\Phi(x,\kappa) = \Psi(x,\kappa)T(\kappa), \quad T(\kappa) = \begin{pmatrix} a(\kappa) & -\tilde{b}(\kappa) \\ b(\kappa) & \tilde{a}(\kappa) \end{pmatrix} \tag{8}$$

where $\psi(x,\kappa)$, $\phi(x,\kappa)$, and $a(\kappa)$ are analytic in the domain of $\mathrm{Im}\,\kappa^{-2} > 0$, namely, in II and IV quadrants in κ-plane; $\tilde{\psi}(x,\kappa)$, $\tilde{\phi}(x,\kappa)$, and $\tilde{a}(\kappa)$ are analytic in the domain of $\mathrm{Im}\,\kappa^{-2} < 0$, namely, in I and III quadrants in κ-plane.

The Jost solutions of DNLS equation have some properties similar to those of NLS equation, for example,

$$\tilde{\psi}(x,\bar{\kappa}) = -i\sigma_2\overline{\psi(x,\kappa)}, \qquad \tilde{\phi}(x,\bar{\kappa}) = i\sigma_2\overline{\phi(x,\kappa)} \tag{9}$$

and

$$\tilde{a}(\bar{\kappa}) = \overline{a(\kappa)}, \qquad \tilde{b}(\bar{\kappa}) = -\overline{b(\kappa)} \tag{10}$$

At the same time, Jost solutions in DNLS equation have some special, that is

$$\Psi(-\kappa) = \sigma_3\Psi(\kappa)\sigma_3, \quad \Phi(-\kappa) = \sigma_3\Phi(\kappa)\sigma_3 \tag{11}$$

and

$$a(-\kappa) = a(\kappa), \quad b(-\kappa) = -b(\kappa) \tag{12}$$

These transformation properties of the Jost solutions and monodramy matrix are essential to solving DNLS equation by IST method, and they are same with that of the spectrum parameter λ. And in the reflectionless case, one should only consider the discreet spectrum part of $a(\kappa)$, that is

$$a(\kappa) = \prod_n \frac{\kappa^2 - \kappa_n^2}{\kappa^2 - \bar{\kappa}_n^2} \tag{13}$$

where κ_n are zeroes of $a(\kappa)$ and the transformation property (12) has been taken into account.

Then also from the first Lax equation (4), one can find the relation between the Jost solutions and the soliton solution u of DNLS equation, for example

$$u = -i \lim_{|\kappa|\to 0} \frac{\kappa^{-1}\tilde{\psi}_2(x,t,\kappa)}{\tilde{\psi}_1(x,t,\kappa)} \tag{14}$$

In the case of reflectionless, one constructs the IST equation by standard procedure in complex κ-plane, that is

$$\tilde{\psi}(x,t,\kappa) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ip(\kappa)} + \sum_n \frac{1}{\kappa - \kappa_n} \frac{1}{\dot{a}(\kappa_n)} \phi(x,t,\kappa_n) e^{ip(\kappa_n)} e^{-ip(\kappa)}, \tag{15}$$

where $p(\kappa) \equiv \kappa^{-2}(x + 2\kappa^{-2}t)$ implying the time dependence, and $\dot{a}(\kappa_n) = \frac{d}{d\kappa} a(\kappa)|_{\kappa=\kappa_n}$. Similarly, one has another IST equation

$$\tilde{\phi}(x,t,\kappa) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ip(\kappa)} + \sum_n \frac{1}{\kappa - \kappa_n} \frac{1}{\dot{a}(\kappa_n)} \psi(x,t,\kappa_n) e^{-ip(\kappa_n)} e^{ip(\kappa)}, \tag{16}$$

These two inverse scattering transformation equations could be taken together to obtain the explicit expressions of the Jost solutions, and then the soliton solution u from (14).

3. INTRODUCTION OF RIGHT INVERSE OF $\tilde{F}(x, t, \zeta)$

The Jost solutions $\Psi(x,t,\kappa)$ and $\Phi(x,t,\kappa)$ are defined by their asymptotic behaviors in (6) and (7). In the complex κ-plane, one should combined the Jost solutions by their analytical properties according to κ, that is

$$\tilde{F}(x,t,\kappa) = (\tilde{\psi}(x,t,\kappa) \;\; \tilde{\phi}(x,t,\kappa)) \tag{17}$$

$$F(x,t,\kappa) = (\phi(x,t,\kappa) \;\; \psi(x,t,\kappa)) \tag{18}$$

and then for discreet spectrum parameter

$$\tilde{F}(x,t,\bar{\kappa}_n) = \tilde{\psi}(x,t,\bar{\kappa}_n)(1 \;\; \tilde{b}_n) \tag{19}$$

$$F(x,t,\kappa_n) = \psi(x,t,\kappa_n)(b_n \;\; 1) \tag{20}$$

In order to demonstrate the Jost solutions satisfy the Lax equations, one should find the inverse of $\tilde{F}(x,t,\zeta)$.

Introducing

$$\tilde{F}(x,t,\kappa)' = \sigma_2 F(x,t,\kappa)^{\mathrm{T}}\sigma_2 \tag{21}$$

$\tilde{F}(x,t,\kappa)\tilde{F}(x,t,\kappa)'$ seems to have two kinds of singularities in the whole κ-plane, one at κ_n and the other at $\bar{\kappa}_n$. But it is not the truth because $\tilde{F}(x,t,\kappa)\tilde{F}(x,t,\kappa)'$ tends to

$$\lim_{\kappa\to\kappa_n}\left\{\frac{1}{\kappa-\kappa_n}\frac{1}{\dot{a}(\kappa_n)}\psi(x,t,\kappa_n)(b_n \quad 1)\right\}\begin{pmatrix}1\\ -b_n\end{pmatrix}\psi(x,t,\kappa_n)^{\mathrm{T}}(-i\sigma_2) = 0 \tag{22}$$

at κ_n and also tends to

$$\lim_{\kappa\to\bar{\kappa}_m}\tilde{\psi}(x,t,\bar{\kappa}_m)(1 \quad -\tilde{b}_m)\left\{\frac{1}{\kappa-\bar{\kappa}_m}\frac{1}{\dot{\tilde{a}}(\bar{\kappa}_m)}\begin{pmatrix}\tilde{b}_m\\ 1\end{pmatrix}\tilde{\psi}(x,\bar{\kappa}_m)^{\mathrm{T}}(i\sigma_2)\right\} = 0. \tag{23}$$

at $\bar{\kappa}_m$, which means that $\tilde{F}(x,t,\kappa)\tilde{F}(x,t,\kappa)'$ is analytic in the whole κ-plane. Then from (15) and (17), the asymptotic behavior of $\tilde{F}(x,t,\kappa)\tilde{F}(x,t,\kappa)'$ is

$$\lim_{|\kappa|\to\infty}\tilde{F}(x,t,\kappa)\tilde{F}(x,t,\kappa)' = I \tag{24}$$

Thus, according to Liouville theorem, $\tilde{F}(x,t,\kappa)'$ is the right inverse of $\tilde{F}(x,t,\kappa)$, that is

$$\tilde{F}(x,t,\kappa)\tilde{F}(x,t,\kappa)' = I \tag{25}$$

And since $\tilde{F}(x,t,\kappa)$ is 2×2 matrix, $\tilde{F}(x,t,\zeta)'$ is also the left inverse of $\tilde{F}(x,t,\zeta)$, which means that $\tilde{F}(x,t,\zeta)'$ is actually the inverse of $\tilde{F}(x,t,\zeta)$.

With Equations (15) and (17), one rewrites $\tilde{F}(x,t,\kappa)$ in the form

$$\tilde{F}(x,t,\kappa) = \tilde{D}(x,t,\kappa)e^{-ip(\kappa)\sigma_3}, \tag{26}$$

where

$$\tilde{D}(x,t,\kappa) = I + \sum_n \frac{1}{\kappa-\kappa_n}\frac{1}{\dot{a}(\kappa_n)}\psi(x,t,\kappa_n)(b_n \quad 1)e^{ip(\kappa_n)\sigma_3} \tag{27}$$

Combining the contributions from κ_n and from $-\kappa_n$, $\tilde{D}(x,t,\kappa)$ becomes

$$\tilde{D}(x,t,\kappa) = I + \sum_n \frac{2\kappa}{\kappa^2-\kappa_n^2}\frac{1}{\dot{a}(\kappa_n)}\begin{pmatrix}0 & \psi_1(x,t,\kappa_n)\\ \psi_2(x,t,\kappa_n)b_n & 0\end{pmatrix}e^{ip(\kappa_n)\sigma_3} \tag{28}$$

$$+\sum_n \frac{2\kappa_n}{\kappa^2-\kappa_n^2}\frac{1}{\dot{a}(\kappa_n)}\begin{pmatrix}\psi_1(x,t,\kappa_n)b_n & 0\\ 0 & \psi_2(x,t,\kappa_n)\end{pmatrix}e^{ip(\kappa_n)\sigma_3}$$

Expanding Equation (30) in the limit of $\kappa \to 0$, one has

$$\tilde{D}(x,t,\kappa) = \sum_{j=0}^{\infty} \nu_j \kappa^j, \tag{29}$$

where

$$\nu_0 = I - \sum_n \frac{2}{\kappa_n} \frac{1}{\dot{a}(\kappa_n)} \begin{pmatrix} \psi_1(x,t,\kappa_n) b_n & 0 \\ 0 & \psi_2(x,t,\kappa_n) \end{pmatrix} e^{ip(\kappa_n)\sigma_3} \tag{30}$$

$$\nu_1 = -\sum_n \frac{2}{\kappa_n^2} \frac{1}{\dot{a}(\kappa_n)} \begin{pmatrix} 0 & \psi_1(x,t,\kappa_n) \\ \psi_2(x,t,\kappa_n) b_n & 0 \end{pmatrix} e^{ip(\kappa_n)\sigma_3} \tag{31}$$

$$\cdots \qquad \cdots$$

Since ν_0 is diagonal and ν_1 is not, one has

$$-i[\nu_0, \sigma_3] = 0, \quad -i[\nu_1, \sigma_3] = U \tag{32}$$

where the relation between the Jost solutions and the soliton solutions (14) has been used.

Then from Equation (23), $\tilde{F}(x,t,\kappa)'$ has the form

$$\tilde{F}(x,t,\kappa)' = e^{ip(\kappa)\sigma_3} \tilde{D}(x,t,\kappa)' \tag{33}$$

where

$$\tilde{D}(x,t,\kappa)' = \sum_{j=0}^{\infty} \nu_j' \kappa^j, \tag{34}$$

and, for $\tilde{D}(x,t,\kappa)\tilde{D}(x,t,\kappa)' = I$, there are

$$\nu_0 \nu_0' = I, \quad \sum_{j+k=m} \nu_j \nu_k' = \delta_{m0} I \tag{35}$$

In the following work, one should show that $\tilde{F}(x,t,\kappa)$ satisfies the Lax equations.

4. DEMONSTRATION OF THE FIRST LAX EQUATION

In order to show that $\tilde{F}(x,t,\kappa)$ satisfies the first Lax equation, one should consider

$$\tilde{F}(x,t,\kappa)_x \tilde{F}(x,t,\kappa)' = \tilde{D}(\kappa)_x \tilde{D}(\kappa)' - i\kappa^{-2} \tilde{D}(\kappa)\sigma_3 \tilde{D}(\kappa)' \tag{36}$$

where $-i\kappa^{-2}\tilde{D}(\kappa)\sigma_3\tilde{D}(\kappa)'$ has obviously a pole of second degree at $\kappa = 0$, and κ_n and $\bar{\kappa}_m$ are not singularities, since the partial derivative of x does not affect the factors $(b_n \;\; 1)(1 \;\; -b_n)^{\mathrm{T}}$ and $(1 \;\; -\tilde{b}_m)(\tilde{b}_m \;\; 1)^{\mathrm{T}}$ in (23) and (24).

Thus, $\tilde{F}(x,t,\kappa)_x\tilde{F}(x,t,\kappa)'$ has the following asymptotic expansion

$$\tilde{F}(x,t,\kappa)_x\tilde{F}(x,t,\kappa)' = R_2\kappa^{-2} + R_1\kappa^{-1} + O(1) \tag{37}$$

as $|\kappa| \to 0$. From Equations (14) and (31), etc., R_2 and R_1 are expressed as

$$R_2 = -i\nu_0\sigma_3\nu_0' = -i\sigma_3, \tag{38}$$

$$R_1 = -i(\nu_1\sigma_3\nu_0' + \nu_0\sigma_3\nu_1') = -i[\nu_1,\sigma_3]\nu_0' = U \tag{39}$$

In the limit of $|\kappa| \to \infty$, from the definition of $\tilde{D}(x,t,\kappa)$ in (28), one has

$$\tilde{D}(x,t,\kappa) \to I, \quad \tilde{D}(x,t,\kappa)_x \to 0 \tag{40}$$

which means that $\tilde{F}(x,t,\kappa)_x\tilde{F}(x,t,\kappa)' - R_2\kappa^{-2} - R_1\kappa^{-1}$ is analytic in whole κ-plane and goes to zero as $|\kappa| \to \infty$. Therefore, by Liouville theorem, one finally obtains

$$\tilde{F}(x,t,\kappa)_x = \{-i\kappa^{-2}\sigma_3 + \kappa^{-1}U\}\tilde{F}(x,t,\kappa) \tag{41}$$

It is just the first Lax equation.

5. DEMONSTRATION OF THE SECOND LAX EQUATION

To derive the second Lax equation, one considers similar to Equation (37)

$$\tilde{F}(x,t,\kappa)_t\tilde{F}(x,t,\kappa)' = \tilde{D}(\kappa)_t\tilde{D}(\kappa)' - i2\kappa^{-4}\tilde{D}(\kappa)\sigma_3\tilde{D}(\kappa)' \tag{42}$$

where $-i2\kappa^{-4}\tilde{D}(\kappa)\sigma_3\tilde{D}(\kappa)^{-1}$ has obviously a pole of fourth degree at $\kappa = 0$, and, similar to (38), there is

$$\tilde{F}(x,t,\kappa)_t\tilde{F}(x,t,\kappa)' = -i2\sigma_3\kappa^{-4} + 2U\kappa^{-3} + S_2\kappa^{-2} + S_1\kappa^{-1} + O(1) \tag{43}$$

as $|\kappa| \to 0$, where

$$S_2 = -i2(\nu_2\sigma_3\nu_0' + \nu_1\sigma_3\nu_1' + \nu_0\sigma_3\nu_2') \tag{44}$$

$$S_1 = -i2(\nu_3\sigma_3\nu_0' + \nu_2\sigma_3\nu_1' + \nu_1\sigma_3\nu_2' + \nu_0\sigma_3\nu_3') \tag{45}$$

The desired first Lax equation (41) is also

$$\tilde{D}(\kappa)_x\tilde{D}(\kappa)' - i\kappa^{-2}\tilde{D}(\kappa)\sigma_3\tilde{D}(\kappa)' = -i\kappa^{-2}\sigma_3 + \kappa^{-1}U \tag{46}$$

and should be expanded in the limit of $|\kappa| \to 0$, of which the terms of power of κ^0 and κ^1 give

$$S_2 = -2\nu_{0x}\nu_0', \qquad S_1 = -2(\nu_{1x}\nu_0' + \nu_{0x}\nu_1') \tag{47}$$

Also from Equation (41), one has

$$\tilde{D}_{xx}(\kappa)\sigma_3\tilde{D}(\kappa)^\dagger - i2\kappa^{-2}\tilde{D}_x(\kappa)\tilde{D}(\kappa)^\dagger = \{\kappa^{-2}U^2 + \kappa^{-1}U_x\}\tilde{D}(\kappa)\sigma_3\tilde{D}(\kappa)^\dagger \tag{48}$$

Correspondingly, its terms of power of κ^{-2} and of κ^{-1} are

$$-i2\nu_{0x}\nu_0' = U^2(\nu_0\sigma_3\nu_0') = U^2\sigma_3 \tag{49}$$

$$-i2(\nu_{1x}\nu_0' + \nu_{0x}\nu_1') = U^2(\nu_1\sigma_3\nu_0' + \nu_0\sigma_3\nu_1') + U_x(\nu_0\sigma_3\nu_0') = iU^3 + U_x\sigma_3 \tag{50}$$

so that one could express S_2 and S_1 in terms of U.

Also for the definition of $\tilde{D}(x,t,\kappa)$ in (28), there is

$$\tilde{D}(x,t,\kappa)_t \to 0, \quad \text{as } |\kappa| \to \infty \tag{51}$$

and thus

$$\partial_t\tilde{F}(x,t,\kappa)\tilde{F}(x,t,\kappa)^{-1} - \{S_4\kappa^{-4} + S_3\kappa^{-3} + S_2\kappa^{-2} + S_1\kappa^{-1}\} \tag{52}$$

is analytic in whole κ-plane and tends to zero in the limit of $|\kappa| \to \infty$. Then from Liouville theorem, it is equal to zero, that is

$$\partial_t\tilde{F}(x,t,\kappa) = \{-i2\kappa^{-4}\sigma_3 + 2\kappa^{-3}U - i\kappa^{-2}U^2\sigma_3 - \kappa^{-1}(-U^3 + iU_x\sigma_3)\}\tilde{F}(x,t,\kappa) \tag{53}$$

which just coincides with the second Lax equation.

6. DISCUSSION

The demonstration of multi-soliton solutions is the fundamental problem of the inverse scattering transformation method for solving nonlinear equations. Equations (41) and (58) show that the Jost solutions obtained by IST in reflectionless case indeed satisfy the two Lax equations, so that, by the compatibility condition, the soliton solutions obtained by the IST method satisfy the DNLS equation actually. It should be noted that all this demonstration is easily done with the inverse scattering spectrum parameter κ, since the residues of $\tilde{F}(x,t,\kappa)_x\tilde{F}(x,t,\kappa)'$ and $\tilde{F}(x,t,\kappa)_t\tilde{F}(x,t,\kappa)'$ in the limit of $|\kappa| \to \infty$ are zero.

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